

Fourier-Motzkin Elimination

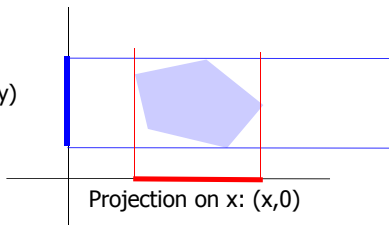
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Fourier-Motzkin Elimination

- Fourier, 1826/1827
- Motzkin, 1938
- **Method:** successive projection of a polyhedron in n -dimensional space into a vector space of dimension $n-1$ by elimination of one variable.

Projection on y : $(0,y)$



Projection on x : $(x,0)$

Fourier-Motzkin Elimination: an example

min/max $x_1 + 3x_2$

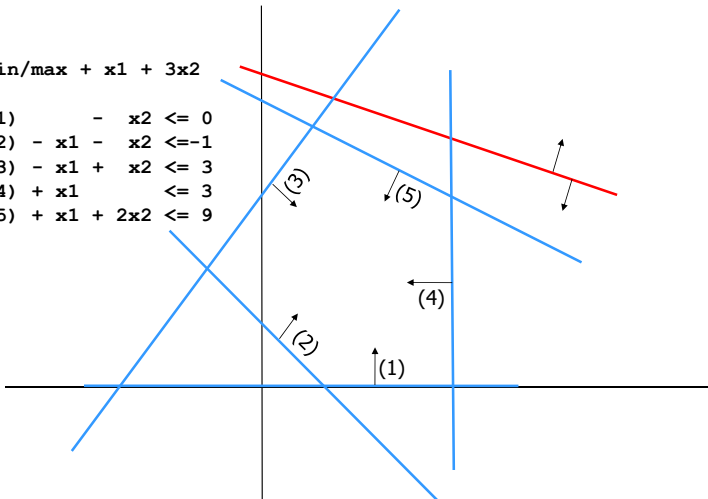
(1) $-x_2 \leq 0$

(2) $-x_1 - x_2 \leq -1$

(3) $-x_1 + x_2 \leq 3$

(4) $+x_1 \leq 3$

(5) $+x_1 + 2x_2 \leq 9$



Fourier-Motzkin Elimination: an example, call of PORTA (Polymake)

DIM = 3

min/max + x1 + 3x2

INEQUALITIES_SECTION

- (1) - x2 <= 0
- (2) - x1 - x2 <=-1
- (3) - x1 + x2 <= 3
- (4) + x1 <= 3
- (5) + x1 + 2x2 <= 9

- (1) - x2 <= 0
- (2) - x1 - x2 <=-1
- (3) - x1 + x2 <= 3
- (4) + x1 <= 3
- (5) + x1 + 2x2 <= 9
- (6) + x1 + 3x2 - x3 <= 0
- (7) - x1 - 3x2 + x3 <= 0



ELIMINATION_ORDER

1 0 0



Fourier-Motzkin Elimination: an example

DIM = 3



DIM = 3

INEQUALITIES_SECTION

(1) (1) - x2 <= 0
 (2,4) (2) - x2 <= 2
 (2,5) (3) + x2 <= 8
 (2,6) (4) +2x2 - x3 <= -1
 (3,4) (5) + x2 <= 6
 (3,5) (6) + x2 <= 4
 (3,6) (7) +4x2 - x3 <= 3
 (7,4) (8) -3x2 + x3 <= 3
 (7,5) (9) - x2 + x3 <= 9
 (7,6)

INEQUALITIES_SECTION

(1) - x2 <= 0
 (2) - x1 - x2 <= -1
 (3) - x1 + x2 <= 3
 (4) + x1 <= 3
 (5) + x1 + 2x2 <= 9
 (6) + x1 + 3x2 - x3 <= 0
 (7) - x1 - 3x2 + x3 <= 0

ELIMINATION_ORDER

1 0 0



Fourier-Motzkin Elimination: an example

DIM = 3



INEQUALITIES_SECTION

(1) (1) - x2 <= 0
 (2,4) (2) - x2 <= 2
 (2,5) (3) + x2 <= 8
 (2,6) (4) +2x2 - x3 <= -1
 (3,4) (5) + x2 <= 6
 (3,5) (6) + x2 <= 4
 (3,6) (7) +4x2 - x3 <= 3
 (7,4) (8) -3x2 + x3 <= 3
 (7,5) (9) - x2 + x3 <= 9
 (7,6)

(1,4) (1) -x3 <= -1
 (1,7) (2) -x3 <= 3
 (2,4) (3) -x3 <= 3
 (2,7) (4) -x3 <= 11
 (8,3) (5) +x3 <= 27
 (8,4) (6) -x3 <= 3
 (8,5) (7) +x3 <= 21
 (8,6) (8) +x3 <= 15
 (8,7) (9) +x3 <= 21
 (9,3) (10) +x3 <= 17
 (9,4) (11) +x3 <= 17
 (9,5) (12) +x3 <= 15
 (9,6) (13) +x3 <= 13
 (9,7) (14) +3x3 <= 39

ELIMINATION_ORDER

0 1 0

min = 1 <= x3 <= 13 = max

x1 = 1

x2 = 0

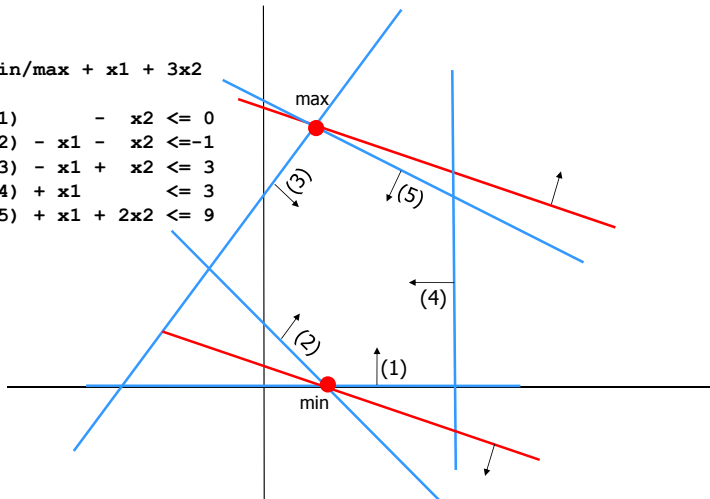
x1 = 1

x2 = 4

Fourier-Motzkin Elimination: an example

min/max $x_1 + 3x_2$

- (1) $-x_2 \leq 0$
- (2) $-x_1 - x_2 \leq -1$
- (3) $-x_1 + x_2 \leq 3$
- (4) $+x_1 \leq 3$
- (5) $+x_1 + 2x_2 \leq 9$



Fourier-Motzkin Elimination

- FME is a wonderful constructive proof method.
- Elimination of all variables of a given inequality system directly yields the **Farkas Lemma**:

$Ax \leq b$ has a solution or $y^T A = 0, y^T b < 0$ has a solution but not both.

- FME is computationally lousy.

Cụ thể về thuật toán và các chứng minh có thể tham khảo tại

- ▶ D. Bertsimas, J. N. Tsitsiklis, and J. Tsitsiklis (1997), Introduction to Linear Optimization, Athena Scientific.
- ▶ A. Schrijver (1998), Theory of Linear and Integer Programming, Wiley, New York.